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Rational solutions of integrable equations via nonlinear superposition formulae

Xing-Biao Hu
CCAST (World Laboratory) PO Box 8730, Beijing 100080, People's Republic of China
State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China

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Abstract. In this paper, a method is developed to obtain rational solutions of integrable equations which include KdV, Boussinesq, KP, Ito, $(1+1)$-dimensional CDGKS, Ramani and $(2+1)$-dimensional CDGKS equations. The key step that we use to find rational solutions of these equations is the use of nonlinear superposition formulae. All these rational solutions obtained for each equation are connected by a Bäcklund transformation, which enables us to find other new solutions through the nonlinear superposition of rational solutions and other known solutions.

1. Introduction

As is known, it is of both theoretical and practical value to search for rational solutions of integrable equations. In theory, it will greatly help to formulate a criterion for integrability as the existence of an infinite sequence of rational solutions appears to be equivalent to having the Painleve property shared by integrable equations [1]. In practice, the rational solutions are of, at least, potential value in physical applications. The question of rational solutions of integrable equations has been discussed widely in the literature since the late 1970's, and different methods have been developed for the construction of rational solutions. For example, the rational solutions of the KdV equation were first discovered by Airault *et al* [2]. Further results and variants were obtained by Adler and Moser [3], Krichever [4], Manakov *et al* [5], Ablowitz and Satsuma [6–8], Kyoto group [9, 10], Matveev [11], Nimmo and Freeman [12], Segal and Wilson [13], Satsuma and Ishimori [14], Gilson and Nimmo [15], Nakamura [16], Pelinovsky and Stepanyants [17, 18] and so on (see [1, 19–23] and references therein).

In this paper, we would like to seek rational solutions of integrable equations by using the Hirota method and Bäcklund transformations (BTs). The key step we employ to find rational solutions of integrable equations is the use of nonlinear superposition formulae. As we will see below, in most cases (especially in $(1+1)$ dimensions) the rational solutions considered are connected by BTs without parameters, while in the literature soliton solutions are connected by BTs with parameters. Bäcklund parameters make it easy to derive nonlinear superposition formulae. For example, using permutability of BTs, one can expect to find corresponding nonlinear superposition formulae. Although permutability of BTs in general still remains an open problem, it provides us with a hint of how to find superposition formulae, and at this stage the only thing left to do is just to prove them directly. However, in the rational solution case, a general procedure to find superposition formulae seems to be unclear. Besides, there is another point to be considered. In the soliton solution case, once we have established a nonlinear superposition formula for them, it is clear that we can systematically construct N-soliton solutions from 1-soliton solutions step by step. Unfortunately, in the rational solution case, there is no such general procedure available to produce a sequence of rational solutions from corresponding superposition formulae.

This paper is organized as follows. Section 2 is devoted to the KP equation. We give two different nonlinear superposition formulae to produce well known rational solutions. However, using these results, we also obtain some new solutions of the KP equation. We treat less well-studied equations such as Ito, $(1+1)$-dimensional CDGKS, Ramani and $(2+1)$-dimensional CDGKS equation in sections 3–6. Rational solutions of these equations are derived by BTs and superposition formulae. To our knowledge, the nonlinear superposition formulae (3.3), (4.3) and (5.3) for the Ito, $(1+1)$-dimensional CDGKS and Ramani equations are new. Conclusions and discussion are given in section 7. Finally, we give three appendices. In appendix A, we list some bilinear operator identities which are used in the paper. In appendix B, we give well known rational solutions to well studied equations: KdV and Boussinesq equations using BTs and nonlinear superposition formulae. It is of interest to note that these rational solutions obtained are connected by BTs which are special cases of more general BTs with parameters. Thus it enables us to obtain other solutions e.g. solutions through superposition of soliton solutions and rational solutions. It is noted that these kind of solutions which are a mix of exponentials and rational expressions have been considered to some extent previously; for instance Matveev's 'positons' [22] have both an algebraic and exponential character although they have not been very widely investigated. The proof details of proposition 2.2 are given in appendix C.

2. The KP equation

The KP equation is given by

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0. \tag{2.1}$$

By the dependent variable transform $u = 2(\ln f)_{xx}$, equation (2.1) is written in bilinear form [24] as

$$(D_x D_t + D_x^4 + \alpha D_y^2) f \cdot f = 0 \tag{2.2}$$

where the bilinear operator $D_x^m D_t^n$ is defined as [24–26]

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t) b(x',t')|_{x'=x,t'=t}.$$

A BT for (2.2) is [27]

$$(aD_y + D_x^2 + \lambda D_x) f \cdot f' = 0 \qquad (D_t + D_x^3 - 3a\lambda D_y - 3aD_x D_y) f \cdot f' = 0 \tag{2.3}$$

where $a^2 = \frac{1}{3}\alpha$ and λ is an arbitrary constant. We represent (2.3) symbolically by $f \xrightarrow{\lambda} f'$. Concerning (2.2), we have the following result [27].

Proposition 2.1. Let f_0 be a solution of (2.2). Suppose that f_1 and f_2 are two solutions of (2.2) such that $f_0 \xrightarrow{\lambda_i} f_i$ $(i = 1, 2)$ and $f_j \neq 0$ $(j = 0, 1, 2)$. Then f_{12} defined by

$$f_0 f_{12} = c[D_x + \tfrac{1}{2}(\lambda_2 - \lambda_1)] f_1 \cdot f_2 \qquad \text{(where } c \text{ is a nonzero constant)}$$

is a new solution of (2.2) which is related to f_1 and f_2 under the BT (2.3) with parameters λ_2 and λ_1 respectively.

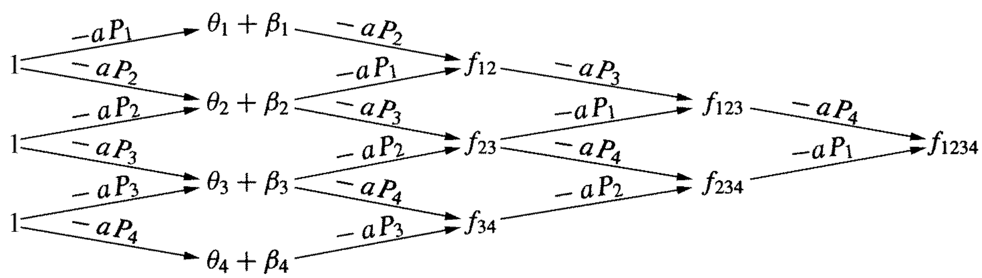
It is noted that in [7] Satsuma and Ablowitz have obtained so-called multi-lump solutions of the KP equation by taking limits of the corresponding soliton solutions. Here we shall use Proposition 2.1 to rederive these solutions. Set $\theta_i = x + P_i y - \alpha P_i^2 t$. It is easily verified that $1 \stackrel{-aP_i}{\to} \theta_i + \beta_i$ (where β_i is a constant). Using Proposition 2.1, we know that

$$f_{12} = \frac{2}{a(P_1 - P_2)}[D_x + \tfrac{1}{2}(-aP_2 + aP_1)](\theta_1 + \beta_1) \cdot (\theta_2 + \beta_2)$$
$$= \theta_1\theta_2 + \left(\beta_1 + \frac{2}{a(P_1 - P_2)}\right)\theta_2 + \left(\beta_2 - \frac{2}{a(P_1 - P_2)}\right)\theta_1 + \beta_1\beta_2 + \frac{2(\beta_2 - \beta_1)}{a(P_1 - P_2)}$$

is a solution of (2.2). If $\alpha = -1$, $P_2 = P_1^*$, $\beta_1 = -2/a(P_1 - P_1^*)$ and $\beta_2 = 2/a(P_1 - P_1^*)$, then we can obtain a so-called one-lump solution

$$f_{12} = \theta_1\theta_1^* - \frac{12}{(P_1 - P_1^*)^2}.$$

Further, using Proposition 2.1, we have



If $\alpha = -1$, $P_3 = P_1^*$, $P_4 = P_2^*$ and β_i $(i = 1, 2, 3, 4)$ are suitably chosen, then we can obtain a so-called two-lump solution from f_{1234}. In principle, along this line, we can obtain multiple-lump solutions of the KP equation (2.2).

Now, we want to seek another series of polynomial solutions of (2.2). First we give the following result.

Proposition 2.2. Suppose f_0, f_1 and f_{12} are three solutions of (2.2). $f_0 \stackrel{0}{\to} f_1 \stackrel{0}{\to} f_{12}$ and $f_0, f_1 \neq 0$. Then there exists a f_2 determined by

$$D_x f_1 \cdot f_2 = c f_0 f_{12} \qquad \text{(where } c \text{ is a non-zero constant)} \tag{2.4}$$

such that f_2 is a new solution of (2.2) and

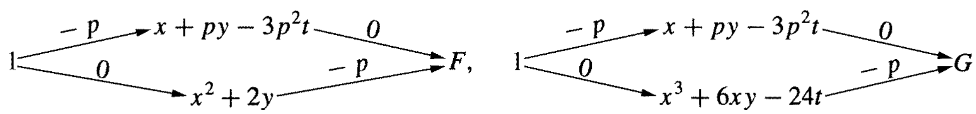
$$f_0 \stackrel{0}{\to} f_2 \stackrel{0}{\to} f_{12}.$$

Remark. In [28], a similar form of superposition formula (2.4) was given without proof for the KdV equation to generate soliton solutions.

Using proposition 2.2, we can obtain polynomial solutions of (2.2). In the following, we choose $a = 1$ for the sake of convenience. In this case, $\alpha = 3$. Now we choose seed solutions $P_0 = 1$, $P_1 = x$. We know $P_0 \stackrel{0}{\to} P_0 \stackrel{0}{\to} P_1$. It is easily verified that $\bar{P}_2 = x^2$ satisfies that $D_x P_0 \cdot \bar{P}_2 = -2P_0 P_1$ and

$$D_y P_0 \cdot \bar{P}_2 - 2D_x P_0 \cdot P_1 = 2P_0^2$$
$$D_t P_0 \cdot \bar{P}_2 - \tfrac{9}{2}D_x^2 P_0 \cdot P_1 + 3D_y P_0 \cdot P_1 + \tfrac{1}{4}D_x^3 P_0 \cdot \bar{P}_2 = 0.$$

We have $k_1(t, y) = 2$, $k_2(t, y) = 0$ and $k(t, y) = 2y$, where functions $k_i(t, y)$ and $k(t, y)$ are introduced in appendix C. Therefore $P_2 = \bar{P}_2 + kP_0 = x^2 + 2y$ is a solution of (2.2) and $P_0 \stackrel{0}{\to} P_2$. Next, using proposition 2.2, we know from $P_0 \stackrel{0}{\to} P_0 \stackrel{0}{\to} P_2$, that $P_3 = x^3 + 6yx - 24t$ is a solution of (2.2) and $P_0 \stackrel{0}{\to} P_3$. In this way, we can deduce a series of polynomial solutions of the KP equation (2.2). Now by combining proposition 2.2 with proposition 2.1, we can proceed to obtain some new rational solutions. For examples, consider equation (2.2) with $\alpha = 3$. From



where $F = \frac{1}{2}p(x^2 + 2y)(x + py - 3p^2t) - x^2 + 2y - 2pxy + 6p^2xt$, $G = \frac{1}{2}p(x + py - 3p^2t)(x^3 + 6xy - 24t) + x^3 + 6xy - 24t - (x + py - 3p^2t)(3x^2 + 6y)$ and p is a constant, we know F and G are rational solutions of (2.2) with $\alpha = 3$. It is also noticed that in [27] Nakamura has obtained new solutions which are superposed by soliton solutions and so-called ripplons. Naturally we can easily obtain new solutions which are superposed by rational solutions and ripplons by combining Nakamura's results and the results obtained here.

3. The Ito equation

The so-called Ito equation in bilinear form is [29, 30]

$$D_t(D_t + D_x^3)f \cdot f = 0. \tag{3.1}$$

A BT for (3.1) is [29, 30]

$$D_xD_tf \cdot f' = 0 \qquad (D_t + D_x^3)f \cdot f' = 0. \tag{3.2}$$

We represent (3.2) symbolically by $f \to f'$. In [30], we have given a nonlinear superposition formula for the Ito equation (3.1) and some particular solutions are presented as an application of the result. In the following, we focus on seeking rational solutions of (3.1). The result is obtained as follows.

Proposition 3.1. Let f_0 and f_1 be two solutions and let $f_0 \to f_1 (f_0, f_1 \neq 0)$. It is assumed that f_2 determined by

$$(D_x^3 - 2D_t)f_1 \cdot f_2 = cf_0^2 \qquad (c \text{ is a constant}) \tag{3.3}$$

satisfies that

$$D_tf_1 \cdot f_2 + \tfrac{1}{4}D_x^3f_1 \cdot f_2 + \tfrac{3}{4}D_x^3f_0 \cdot P = 0 \tag{3.4}$$

where P is determined by the relation

$$D_xf_0 \cdot P = D_xf_1 \cdot f_2. \tag{3.5}$$

Then f_2 is a new solution of (3.1), $f_0 \to f_2$, and

$$D_tf_1 \cdot f_2 + D_tf_0 \cdot P = c_1(t)f_0^2 \tag{3.6}$$

$$D_tf_0 \cdot P + \tfrac{1}{4}D_x^3f_0 \cdot P + \tfrac{3}{4}D_x^3f_1 \cdot f_2 = c_2(t)f_0^2 \tag{3.7}$$

where $c_i(t)$ $(i = 1, 2)$ is some suitable function of t. Furthermore, if we can choose suitable P from (3.5) such that $c_1(t) = c_2(t) = 0$ in (3.6) and (3.7), then P thus obtained is also a new solution of (3.1) and $f_1 \to P$, $f_2 \to P$.

Proof. First, using (3.3)–(3.5), we can show that

$$(D_t + D_x^3) f_0 \cdot f_2 = 0.$$

Next we have from $[(D_t + D_x^3) f_0 \cdot f_1]_x f_2 - [(D_t + D_x^3) f_0 \cdot f_2]_x f_1 = 0$, that

$$D_x[D_t f_0 \cdot P + \tfrac{1}{4} D_x^3 f_0 \cdot P + \tfrac{3}{4} D_x^3 f_1 \cdot f_2] \cdot f_0^2 = 0$$

which implies that (3.7) holds. Using (3.3), (3.4) and (3.7) we can deduce that (3.6) holds, and, further, that $D_x D_t f_0 \cdot f_2 = 0$. Therefore f_2 is a new solution of (3.1) and $f_0 \to f_2$. Finally, if $c_1(t) = c_2(t) = 0$ in (3.6) and (3.7), we can prove that P is also a solution of (3.1) and $f_1 \to P$, $f_2 \to P$ similar to that in [30]. The details are omitted. □

Using proposition 3.1, we can obtain a series of rational solutions of the Ito equation (3.1). In the following, we only give two examples.

Example 1. Choose $f_1 = 1$, $f_0 = x$. It is easily verified that $f_2 = tx + \frac{1}{12}x^4$ satisfies (3.3) with $c = 0$ and (3.4) with $P = t - \frac{1}{6}x^3$. Therefore, $tx + \frac{1}{12}x^4$ is a solution of the Ito equation (3.1) and $x \to tx + \frac{1}{12}x^4$. Futhermore, (3.6) and (3.7) hold with $c_i(t) = 0$. Therefore $t - \frac{1}{6}x^3$ is also a solution of (3.1) and $1 \to t - \frac{1}{6}x^3$, $tx + \frac{1}{12}x^4 \to t - \frac{1}{6}x^3$. Next we choose $f_0 = tx + \frac{1}{12}x^4$, $f_1 = x$. It can be verified that $f_2 = x^{10} + 36x^7t - 1512x^4t^2 - 6048xt^3$ satisfies (3.3)–(3.5) with $c = -54432$ and $P = 36x^7 + (\frac{1}{12}A - 1512)x^4t + Axt^2$ (A is an arbitrary constant). Therefore $x^{10} + 36x^7t - 1512x^4t^2 - 6048xt^3$ is a new polynomial solution of (3.1) and $x^{10} + 36x^7t - 1512x^4t^2 - 6048xt^3 \to tx + \frac{1}{12}x^4$. More generally, it is easily verified that $f_1 = x^\alpha$ and $f_0 = tx^\alpha + (18\alpha - 6)^{x^{\alpha+3}}$ ($\alpha \neq \frac{1}{3}$) are two solutions of the Ito equation (3.1) and $x^\alpha \to tx^\alpha + (18\alpha - 6)^{x^{\alpha+3}}$. A further detailed calculation shows that

$$\begin{aligned} f_2 = {} & x^{\alpha+9} + 9(6\alpha - 2)x^{\alpha+6}t + (9\alpha - 3)(108\alpha - 360)x^{\alpha+3}t^2 + (6\alpha - 2)(9\alpha - 3) \\ & \times (108\alpha - 360)x^\alpha t^3 \end{aligned}$$

satisfies (3.3)–(3.5) with $c = 54(3\alpha - 1)^2(108\alpha - 360)$ and

$$P = 3(18\alpha - 6)x^{\alpha+6} + \left[\frac{A}{18\alpha - 6} + (9\alpha - 3)(108\alpha - 360)\right] x^{\alpha+3}t + Ax^\alpha t^2$$

where A is an arbitrary constant. Therefore

$$\begin{aligned} & x^{\alpha+9} + 9(6\alpha - 2)x^{\alpha+6}t + (9\alpha - 3)(108\alpha - 360)x^{\alpha+3}t^2 + (6\alpha - 2)(9\alpha - 3) \\ & \times (108\alpha - 360)x^\alpha t^3 \end{aligned}$$

is a solution of (3.1) and

$$\begin{aligned} tx^\alpha + \frac{1}{18\alpha - 6}x^{\alpha+3} \to {} & x^{\alpha+9} + 9(6\alpha - 2)x^{\alpha+6}t + (9\alpha - 3)(108\alpha - 360)x^{\alpha+3}t^2 \\ & + (6\alpha - 2)(9\alpha - 3)(108\alpha - 360)x^\alpha t^3. \end{aligned}$$

Example 2. Choose $f_0 = x^9 - 18x^6t + 1080x^3t^2 - 2160t^3$, $f_1 = t - \frac{1}{6}x^3$. From example 1, we know f_0 and f_1 are two solutions of the Ito equation (3.1) and $f_0 \to f_1$. A detailed calculation shows that

$$\begin{aligned} f_2 = {} & x^{18} - 36x^{15}t + 5400x^{12}t^2 - 626400x^9t^3 + 5443200x^6t^4 - 130636800x^3t^5 \\ & + 130636800t^6 \end{aligned}$$

satisfies (3.3)–(3.5) with $c = 420$ and $P = -\frac{5}{6}x^{12} + 180x^6t^2 - 111600x^3t^3 + 324000t^4$. Therefore f_2 is a new polynomial solution of (3.1).

4. The $(1+1)$-dimensional CDGKS equation

The $(1+1)$-dimensional CDGKS equation in bilinear form is [31–33]

$$(D_x^6 - D_x D_t) f \cdot f = 0. \tag{4.1}$$

A BT for (4.1) is [33]

$$D_x^3 f \cdot f' = 0 \qquad (D_t + \tfrac{3}{2} D_x^5) f \cdot f' = 0. \tag{4.2}$$

It is noted that in [34] we have given a nonlinear superposition formula for (4.1) and some particular solutions are obtained as an application of the result. Here we focus on seeking polynomial solutions of (4.1). To this end, let f_0 and f_1 be two solutions of (4.1), $f_0, f_1 \neq 0$, and $f_0 \to f_1$. Suppose f_2 determined by

$$D_x^3 f_1 \cdot f_2 = c f_0^2 \qquad (c \text{ is a constant}) \tag{4.3}$$

satisfies that

$$\tfrac{3}{2} D_t f_0 \cdot P - \tfrac{1}{2} D_t f_1 \cdot f_2 + \tfrac{21}{32} D_x^5 f_1 \cdot f_2 + \tfrac{27}{32} D_x^5 f_0 \cdot P - \tfrac{15}{4} c D_x^2 f_0 \cdot f_0 = k(t) f_0^2 \tag{4.4}$$

where $k(t)$ is some function of t and P is determined by

$$D_x f_1 \cdot f_2 = D_x f_0 \cdot P.$$

Now we have, by using (A5), that

$$\begin{aligned}(D_x^3 f_0 \cdot f_1)(D_x f_2 \cdot f_0) - D_x (D_x^3 f_0 \cdot f_1) \cdot f_2 f_0 - \frac{f_0}{f_1} D_x (D_x^3 f_2 \cdot f_0) \cdot f_1^2 \\ = D_x f_0^2 \cdot (D_x^3 f_2 \cdot f_1) = 0\end{aligned}$$

which implies that

$$D_x (D_x^3 f_0 \cdot f_2) \cdot f_1^2 = 0$$

i.e.

$$D_x^3 f_0 \cdot f_2 = k_1(t) f_1^2 \tag{4.5}$$

where $k_1(t)$ is some function of t. In the following we assume that $k_1(t) = 0$ in (4.5). In this case, we have

$$D_x^3 f_0 \cdot f_2 = 0.$$

Next, it follows from $(D_x^3 f_0 \cdot f_1) f_2 - (D_x^3 f_0 \cdot f_2) f_1 = 0$, that

$$D_x^3 f_0 \cdot P = -\frac{1}{3} D_x^3 f_1 \cdot f_2 = -\frac{c}{3} f_0^2. \tag{4.6}$$

Furthermore, we have, by using (A6)–(A8),

$$[(D_t + \tfrac{3}{2} D_x^5) f_0 \cdot f_2](D_x f_1 \cdot f_0) - D_x [(D_t + \tfrac{3}{2} D_x^5) f_0 \cdot f_2] \cdot f_1 f_0 = 0$$

which implies that

$$-\frac{f_0}{f_1} D_x [(D_t + \tfrac{3}{2} D_x^5) f_0 \cdot f_2] \cdot f_1^2 = 0$$

i.e.

$$(D_t + \tfrac{3}{2} D_x^5) f_0 \cdot f_2 = k_2(t) f_1^2 \tag{4.7}$$

where $k_2(t)$ is some function of t. Finally, if $k_2(t) = 0$, we have

$$(D_t + \tfrac{3}{2} D_x^5) f_0 \cdot f_2 = 0.$$

Thus f_2 is a new solution of (4.1) and $f_0 \to f_2$.

Using the above result, we can obtain polynomial solutions of the $(1+1)$-dimensional CDGKS equation (4.1). For example, choose $f_0 = x$, $f_1 = 1$. A direct calculation shows that $f_2 = 12t + \frac{1}{60}x^5$ determined by (4.3) with $c = -1$ satisfies (4.4) with $P = \frac{1}{36}x^4$ and $k_i(t) = 0$ $(i = 1, 2)$ in (4.5) and (4.7). Therefore $12t + \frac{1}{60}x^5$ is a polynomial solution of (4.1). In general, suppose that two polynomials $P_{(n-1)(3n-4)/2}$ and $P_{n(3n-1)/2}$ are solutions of (4.1), that $P_{n(3n-1)/2} \to P_{(n-1)(3n-4)/2}$, and that

$$\deg(P_{n(3n-1)/2}) = \frac{n(3n-1)}{2} \qquad \deg(P_{(n-1)(3n-4)/2}) = \frac{(n-1)(3n-4)}{2}$$

where the degrees of x and t are defined by

$$\deg(x) = 1 \qquad \deg(t) = 5.$$

Then we can seek a polynomial solution $P_{(n+1)(3n+2)/2}$ $(\deg(P_{(n+1)(3n+2)/2}) = (n+1)(3n+2)/2)$ of (4.1) by using the result obtained above. Thus it is possible to find more polynomial solutions of (4.1). Next we choose $f_0 = x^2$, $f_1 = 1$. A detailed calculation shows that $f_2 = 12x^2t + \frac{1}{210}x^7$ determined by (4.3) with $c = -1$ satisfies (4.4) with $P = -12t + \frac{1}{90}x^5$ and $k(t) = 0$, and $k_i(t) = 0$ $(i = 1, 2)$ in (4.5) and (4.7). Therefore $12x^2t + \frac{1}{210}x^7$ is a polynomial solution of (4.1). In general, suppose that two polynomials $P_{(n-1)(3n-2)/2}$ and $P_{n(3n+1)/2}$ are solutions of (4.1), that $P_{n(3n+1)/2} \to P_{(n-1)(3n-2)/2}$, and that

$$\deg(P_{n(3n+1)/2}) = \frac{n(3n+1)}{2} \qquad \deg(P_{(n-1)(3n-2)/2}) = \frac{(n-1)(3n-2)}{2}.$$

Then we can seek a polynomial solution $P_{(n+1)(3n+4)/2}$ $(\deg(P_{(n+1)(3n+4)/2}) = (n+1)(3n+4)/2)$ of (4.1) by using the result obtained above. Thus it is possible to find more other polynomial solutions of (4.1).

5. The Ramani equation

The so-called Ramani equation is [35]

$$(D_x^6 - 5D_x^3 D_t - 5D_t^2) f \cdot f = 0. \tag{5.1}$$

A BT for (5.1) is [36]

$$(D_x^3 - D_t) f \cdot f' = 0 \qquad (D_x^5 + 5D_x^2 D_t) f \cdot f' = 0. \tag{5.2}$$

We represent (5.2) symbolically by $f \to f'$. Concerning (5.1), we have the following result.

Proposition 5.1. Let f_0 and f_1 be two solutions of (5.1) and let $f_0 \to f_1$ $(f_0, f_1 \neq 0)$. It is assumed that f_2 determined by

$$(D_x^3 + 2D_t) f_1 \cdot f_2 = c f_0^2 \qquad (c \text{ is a constant}) \tag{5.3}$$

satisfies that

$$D_t f_1 \cdot f_2 + \tfrac{1}{4} D_x^3 f_1 \cdot f_2 + \tfrac{3}{4} D_x^3 f_0 \cdot P = 0 \tag{5.4}$$

where P is determined by the relation

$$D_x f_0 \cdot P = D_x f_1 \cdot f_2. \tag{5.5}$$

Then we have

$$(D_x^3 - D_t) f_0 \cdot f_2 = 0 \tag{5.6}$$

$$\tfrac{1}{4} D_x^3 f_0 \cdot P - D_t f_0 \cdot P + \tfrac{3}{4} D_x^3 f_1 \cdot f_2 = c_1(t) f_0^2 \tag{5.7}$$

where $c_1(t)$ is some function of t. Further, if f_2 and P satisfies

$$D_x^5 f_1 \cdot f_2 + 20 D_x^2 D_t f_1 \cdot f_2 + 15 D_x^5 f_0 \cdot P + 60 D_x^2 D_t f_0 \cdot P + 60 c_1(t) D_x^2 f_0 \cdot f_0 = 0 \tag{5.8}$$

then f_2 is a solution of (5.1), $f_0 \to f_2$ and we have

$$D_x^5 f_0 \cdot P + 20 D_x^2 D_t f_0 \cdot P + 15 D_x^5 f_1 \cdot f_2 + 60 D_x^2 D_t f_1 \cdot f_2 + 20 c_1(t) D_x^2 f_0 \cdot f_0 = c_2(t) f_0^2 \tag{5.9}$$

where $c_2(t)$ is some function of t. Moreover, if $c_i(t) = 0$ $(i = 1, 2)$, then P is also a solution of (5.1) and $f_1 \to P$, $f_2 \to P$.

Proof. Similar to the proof of proposition 3.1, we can show that (5.6) and (5.7) hold. Furthermore, a detailed calculation shows that $(D_x^5 + 5 D_x^2 D_t) f_0 \cdot f_2 = 0$ if f_2 and P satisfies (5.8). Moreover, from

$$[(D_x^3 - D_t) f_0 \cdot f_1]_{xx} f_2 - [(D_x^3 - D_t) f_0 \cdot f_2]_{xx} f_1 = 0$$

and

$$\begin{aligned}[(D_x^5 + 5 D_x^2 D_t) f_0 \cdot f_1]_x f_2 - [(D_x^5 + 5 D_x^2 D_t) f_0 \cdot f_2]_x f_1 + 5[(D_x^3 - D_t) f_0 \cdot f_1]_{xxx} f_2 \\ -5[(D_x^3 - D_t) f_0 \cdot f_2]_{xxx} f_1 = 0\end{aligned}$$

we can deduce that (5.9) holds and

$$D_x^5 f_1 \cdot f_2 - 4 D_x^2 D_t f_1 \cdot f_2 - D_x^5 f_0 \cdot P + 4 D_x^2 D_t f_0 \cdot P + 4 c_1(t) D_x^2 f_0 \cdot f_0 = 0.$$

Finally, if $c_i(t) = 0$ $(i = 1, 2)$ in (5.7) and (5.9), we can prove P is a solution of (5.1) and $f_1 \to P$, $f_2 \to P$. □

As an application of proposition 5.1, we can obtain some polynomial solutions of (5.1). For example, choose $f_0 = x$, $f_1 = 1$. It is easily verified that $f_2 = tx - \frac{1}{12}x^4$ satisfies (5.3)–(5.5) and (5.8) with $c = 0$ and $P = t + \frac{1}{6}x^3$. Therefore $tx - \frac{1}{12}x^4$ is a solution of the Ramani equation (5.1) and $x \to tx - \frac{1}{12}x^4$. Futhermore, (5.7) and (5.9) hold with $c_i(t) = 0$. Therefore $t + \frac{1}{6}x^3$ is also a solution of (5.1). Next we choose $f_0 = -tx + \frac{1}{12}x^4$, $f_1 = x$. It can be verified that $f_2 = x^{10} - 36x^7 t - 1512 x^4 t^2 + 6048 x t^3$ satisfies (5.3)–(5.5) and (5.8) with $c = -54432$, $P = 36x^7 + (1512 - \frac{1}{12}A)x^4 t + Axt^2$ (A is an arbitrary constant) and $c_1(t) = -A - 18144$. Therefore $x^{10} - 36x^7 t - 1512 x^4 t^2 + 6048 x t^3$ is a new polynomial solution of (5.1) and $x^{10} - 36x^7 t - 1512 x^4 t^2 + 6048 x t^3 \to -tx + \frac{1}{12}x^4$. In particular, when $A = -18144$, we have $c_1(t) = c_2(t) = 0$. Therefore $P = 36x^7 + 3024 x^4 t - 18144 x t^2$ is also a polynomial solution of (5.1)

6. The $(2+1)$-dimensional CDGKS equation

The $(2+1)$-dimensional CDGKS equation in bilinear form is [37, 38]

$$(D_x^6 - 5 D_x^3 D_y - 5 D_y^2 + 9 D_x D_t) f \cdot f = 0. \tag{6.1}$$

A BT for (6.1) is [39]

$$(D_x^3 - D_y - 3k D_x^2 + 3k^2 D_x) f \cdot f' = 0 \tag{6.2a}$$

$$(-D_x^5 - 5 D_x^2 D_y + 5k D_x^4 - 5k^2 D_x^3 - 10k^2 D_y + 10k D_x D_y + 6 D_t) f \cdot f' = 0 \tag{6.2b}$$

where k is an arbitrary constant. In what follows, we represent (6.2) symbolically by $f \xrightarrow{k} f'$. Let f_0 be a solution of (6.1), $f_0 \neq 0$. Suppose that f_i $(i = 1, 2)$ is a solution of

(6.1) which is related to f_0 under BT (6.2) with k_i, i.e. $f_0 \xrightarrow{k_i} f_i$ $(i = 1, 2)$. Then we can prove that f_{12} defined by

$$[D_x - (k_1 + k_2)]f_0 \cdot f_{12} = [D_x + (k_1 - k_2)]f_1 \cdot f_2 \tag{6.3}$$

is a solution of (6.1) under certain conditions. The details are given in [39]. Using nonlinear superposition formula (6.3), we can derive some polynomial solutions, soliton solutions and other solutions of (6.1). Here we just give an example of a solution of (6.1). We choose $f_0 = 1$, $f_i = \theta_i + \beta_i \equiv x + 3k_i^2 y + 5k_i^4 t + \beta_i$ (β_i is a constant, $i = 1, 2$). It is easily verified that 1 and $\theta_i + \beta_i$ are two solutions of (6.1) and $1 \xrightarrow{k_i} \theta_i + \beta_i$. Thus from (6.3), we can obtain

$$\begin{aligned} f_{12} = \frac{k_2 - k_1}{k_2 + k_1}\theta_1\theta_2 - \frac{(k_1^2 - k_2^2)\beta_2 - 2k_1}{(k_1 + k_2)^2}\theta_1 - \frac{(k_1^2 - k_2^2)\beta_1 + 2k_2}{(k_1 + k_2)^2}\theta_2 \\ +[(k_1 - k_2)(\beta_1 + \beta_2) - (k_1 + k_2)(\beta_2 - \beta_1) + (k_2^2 - k_1^2)\beta_1\beta_2 \\ +2(k_2 - k_1)(k_1 + k_2)][(k_1 + k_2)^2]^{-1}. \end{aligned}$$

It can be verified that f_{12} thus obtained is a polynomial solution of (6.1) and $f_1 \xrightarrow{k_2} f_{12}$, $f_2 \xrightarrow{k_1} f_{12}$. Similar to the KP case, along this line, it is natural to find more polynomial solutions. We finish this section by giving another result for the (2+1)-dimensional CDGKS equation (6.1).

Proposition 6.1. [39]. Let f_0 and f_1 be two solutions of (6.1) and let $f_0 \xrightarrow{0} f_1$ $(f_0, f_1 \neq 0)$. Suppose that there exist f_2 and P such that the following relations hold:

$$D_x f_1 \cdot f_2 = D_x f_0 \cdot P \tag{6.4}$$

$$\tfrac{1}{4}D_x^3 f_1 \cdot f_2 - D_y f_1 \cdot f_2 + \tfrac{3}{4}D_x^3 f_0 \cdot P = 0. \tag{6.5}$$

Then we have

$$(D_x^3 - D_y) f_0 \cdot f_2 = 0 \tag{6.6}$$

$$\tfrac{1}{4}D_x^3 f_0 \cdot P - D_y f_0 \cdot P + \tfrac{3}{4}D_x^3 f_1 \cdot f_2 = c_1(t, y) f_0^2 \tag{6.7}$$

where $c_1(t, y)$ is some function of t and y. Further, if f_2 and P satisfies

$$\begin{aligned} -96D_t f_1 \cdot f_2 + D_x^5 f_1 \cdot f_2 + 20D_x^2 D_y f_1 \cdot f_2 + 15D_x^5 f_0 \cdot P + 60D_x^2 D_y f_0 \cdot P \\ +60c_1(t, y)D_x^2 f_0 \cdot f_0 = 0 \end{aligned} \tag{6.8}$$

then f_2 is a solution of (6.1) and $f_0 \xrightarrow{0} f_2$, and we have

$$\begin{aligned} -96D_t f_0 \cdot P + D_x^5 f_0 \cdot P + 20D_x^2 D_t f_0 \cdot P + 15D_x^5 f_1 \cdot f_2 + 60D_x^2 D_t f_1 \cdot f_2 \\ +20c_1(t, y)D_x^2 f_0 \cdot f_0 = c_2(t, y) f_0^2 \end{aligned} \tag{6.9}$$

where $c_2(t, y)$ is some function of t. Moreover, if $c_i(t, y) = 0$ $(i = 1, 2)$, then P is also a solution of (6.1) and $f_1 \xrightarrow{0} P$, $f_2 \xrightarrow{0} P$.

Using Proposition 6.1, we can also obtain some polynomial solutions of (6.1).

7. Conclusion and discussion

In this paper, a method is developed to obtain rational solutions of integrable equations. We have obtained rational solutions of KdV, Boussinesq, KP, Ito, $(1+1)$-dimensional CDGKS, Ramani and $(2+1)$-dimensional CDGKS equations. There are many methods to obtain rational solutions. Compared with these methods, here emphasis is placed on producing rational solutions via nonlinear superposition formulae and showing that rational solutions are connected by BTs. We have seen that besides well studied equations such as the KdV, Boussinesq and KP, rational solutions of less studied equations such as the Ito and Ramani can be obtained by this method. Furthermore, since rational solutions obtained in this paper are connected by BTs which are special cases of more general BTs, it enables us to obtain other types of solutions which are superposed by rational solutions and some particular solutions, e.g. soliton solutions in the KdV case and ripplon solutions in the KP case and so on. Besides, existence of nonlinear superposition formulae is an interesting topic by itself in soliton theory, and it seems to be reasonable to view existence of nonlinear superposition formulae as one of the common features shared by integrable equations. It is also noticed that very recently there have been attempts to deduce integrable differential-difference and difference equations from nonlinear superposition formulae (see, e.g. [40, 41]).

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Appendix A

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$D_x(D_t a\cdot b)\cdot a^2 = D_t(D_x a\cdot b)\cdot a^2 \tag{A1}$$

$$-abD_x^3 a\cdot b + D_x(D_x^2 a\cdot a)\cdot b^2 = D_x(D_x^2 a\cdot b)\cdot ab - (D_x^2 a\cdot b)(D_x a\cdot b) \tag{A2}$$

$$(D_t a\cdot b)c - (D_t a\cdot c)b = -aD_t b\cdot c \tag{A3}$$

$$(D_x^3 a\cdot b)c - (D_x^3 a\cdot c)b = -3a_{xx}D_x b\cdot c + 3a_x(D_x b\cdot c)_x - \tfrac{1}{4}a[D_x^3 b\cdot c + 3(D_x b\cdot c)_{xx}] \tag{A4}$$

$$D_x a^2\cdot(D_x^3 c\cdot b) = (D_x^3 a\cdot b)(D_x c\cdot a) - D_x(D_x^3 a\cdot b)\cdot ca - \frac{a}{b}D_x(D_x^3 c\cdot a)\cdot b^2 \tag{A5}$$

$$\begin{aligned}&[(D_t + \tfrac{3}{2}D_x^5)a\cdot b](D_x c\cdot a) - D_x[(D_t+\tfrac{3}{2}D_x^5)a\cdot b]\cdot ca - (D_x a\cdot b)[(D_t+\tfrac{3}{2}D_x^5)c\cdot a]\\ &\qquad -D_x ab\cdot[(D_t+\tfrac{3}{2}D_x^5)c\cdot a] = \tfrac{3}{2}D_t a^2\cdot(D_x c\cdot b) - \tfrac{1}{2}D_x a^2\cdot(D_t c\cdot b)\\ &\qquad +\tfrac{3}{2}\{(D_x^5 a\cdot b)(D_x c\cdot a) - (D_x a\cdot b)(D_x^5 c\cdot a)\\ &\qquad -D_x[(D_x^5 a\cdot b)\cdot ca - ab\cdot(D_x^5 c\cdot a)]\}\end{aligned} \tag{A6}$$

$$\begin{aligned}&(D_x^5 a\cdot b)(D_x c\cdot a) - (D_x a\cdot b)(D_x^5 c\cdot a) = -\tfrac{1}{8}D_x^5 a^2\cdot(D_x c\cdot b)\\ &\qquad +\tfrac{1}{8}D_x[a^2\cdot(D_x^5 c\cdot b) + 5(D_x^4 a\cdot a)\cdot(D_x c\cdot b) + 10(D_x^2 a\cdot a)\cdot(D_x^3 c\cdot b)]\end{aligned} \tag{A7}$$

$$\begin{aligned}&D_x[(D_x^5 a\cdot b)\cdot ca - ab\cdot(D_x^5 c\cdot a)] = -\tfrac{1}{16}D_x^5 a^2\cdot(D_x c\cdot b)\\ &\qquad -\tfrac{5}{8}D_x^3[a^2\cdot(D_x^3 c\cdot b) - (D_x^2 a\cdot a)\cdot(D_x c\cdot b)]\\ &\qquad -\tfrac{5}{16}D_x[a^2\cdot(D_x^5 c\cdot b) + 2(D_x^2 a\cdot a)\cdot(D_x^3 c\cdot b) - 3(D_x^4 a\cdot a)\cdot(D_x c\cdot b)].\end{aligned} \tag{A8}$$

Appendix B. The KdV and Boussinesq equations

In this appendix, we shall rederive the well known rational solutions of the KdV and Boussinesq equations using BTs and nonlinear superposition formulae.

The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0. \tag{B1}$$

By the dependent variable transform $u = 2(\ln f)_{xx}$, equation (B.1) is written in bilinear form as

$$D_x(D_t + D_x^3) f \cdot f = 0. \tag{B2}$$

A BT for (B2) is

$$D_x^2 f \cdot f' = 0 \qquad (D_t + D_x^3) f \cdot f' = 0. \tag{B3}$$

We represent (B3) symbolically by $f \to f'$. It is evident that

$$f \to f' \iff f' \to f$$

Note that in [3] Adler and Moser first discovered that rational solutions of (B2) are generated by the following formula:

$$D_x f_{N-1} \cdot f_{N+1} = c f_N^2 \tag{B4}$$

by considering the factorization of the Sturm–Liouville operator. In [6], Ablowitz and Satsuma recovered (B4) by limiting the corresponding nonlinear superposition formula of the KdV soliton solutions. Here we shall establish the nonlinear superposition formula of rational solutions for the KdV equation (B2) directly and rigorously. By means of BT (B3), we have the following result.

Proposition B.1. Suppose f_0 and f_1 are two solutions of (B2) which are connected by (B3), $f_0, f_1 \neq 0$. Then there exists an f_2 determined by

$$D_x f_1 \cdot f_2 = c f_0^2 \qquad \text{(where } c \text{ is a non-zero constant)} \tag{B5}$$

such that f_2 is a new solution of (B2) which is connected with f_0 by BT (B3), i.e. $f_0 \to f_2$.

Proof. First we choose a particular solution $\tilde{f}_2$ from (B5). Thus we have

$$D_x f_1 \cdot \tilde{f}_2 = c f_0^2. \tag{B5$'$}$$

In this case, we have, by using (A1) and (A2),

$$D_x[D_t f_1 \cdot \tilde{f}_2 + 2c D_x^2 f_0 \cdot f_0] \cdot f_1^2 = 0$$

which implies that

$$D_t f_1 \cdot \tilde{f}_2 + 2c D_x^2 f_0 \cdot f_0 = k_1(t) f_1^2$$

where $k_1(t)$ is a suitable function of t. Now we choose $f_2 = \tilde{f}_2 + f_1 \int^t k_1(t')\,\mathrm{d}t'$. It is easily verified that f_2 satisfies (B5) and

$$D_t f_1 \cdot f_2 + 2c D_x^2 f_0 \cdot f_0 = 0 \qquad D_x^3 f_1 \cdot f_2 = -c D_x^2 f_0 \cdot f_0. \tag{B6}$$

Thus we have, by using (A3), (A4) and (B5)–(B6),

$$D_x^2 f_0 \cdot f_2 = 0 \qquad (D_t + D_x^3) f_0 \cdot f_2 = 0.$$

Thus we have completed the proof of proposition B.1. □

In the following, we shall consider the rational solutions of (B2). To this end, we define the degree of x and t as

$$\deg(x) = 1 \qquad \deg(t) = 3.$$

We have the following.

Corollary B.2. Let two polynomials $P_{N(N+1)/2}$ and $P_{(N+1)(N+2)/2}$ be two solutions of (B2) which are connected by BT (B3), and

$$\deg(P_{N(N+1)/2}) = \frac{N(N+1)}{2} \qquad \deg(P_{(N+1)(N+2)/2}) = \frac{(N+1)(N+2)}{2}.$$

Furthermore, suppose that there exists a polynomial $P_{(N+2)(N+3)/2}$ of degree $(N+2)(N+3)/2$ such that

$$D_x P_{N(N+1)/2} \cdot P_{(N+2)(N+3)/2} = c P^2_{(N+1)(N+2)/2} \qquad (c \text{ is a non-zero constant}).$$

Then we have a polynomial $\tilde{P}_{(N+2)(N+3)/2}$ of degree $(N+2)(N+3)/2$ such that $\tilde{P}_{(N+2)(N+3)/2}$ is a solution of (B2) and $P_{(N+1)(N+2)/2} \to \tilde{P}_{(N+2)(N+3)/2}$. In particular, when N is not divided by 3 $\tilde{P}_{(N+2)(N+3)/2} = P_{(N+2)(N+3)/2}$.

Using corollary B.2, we can easily re-obtain a series of rational solutions of (B2). For example, choose $P_0 = 1$, $P_1 = x$. It is evident that P_0 and P_1 are two solutions of (B2) and $P_0 \to P_1$. It is easily verified that $P_3 = x^3$ satisfies

$$D_x P_0 \cdot P_3 = -3P_1^2.$$

Further, we have

$$D_t P_0 \cdot P_3 - 6D_x^2 P_1 \cdot P_1 = 12P_0^2.$$

Therefore $\tilde{P}_3 = P_3 + 12tP_0 = x^3 + 12t$ is a solution of (B2) and $P_1 \to \tilde{P}_3$. Next, it can be verified that $P_6 = x^6 + 60x^3t - 720t^2$ satisfies

$$D_x P_1 \cdot P_6 = -5\tilde{P}_3^2.$$

Therefore P_6 is a solution of (B2) and $\tilde{P}_3 \to P_6$. Furthermore, it can be verified that $P_{10} = x^{10} + 180x^7t + 302400xt^3$ satisfies

$$D_x \tilde{P}_3 \cdot P_{10} = -7P_6^2.$$

Therefore P_{10} is also a solution of (B2) and $P_6 \to P_{10}$. In general, along this line, we can obtain a series of rational solutions of (B2). It is noted that these rational solutions obtained for the KdV equation are connected by BT (B3) which is a special case of the following BT with an arbitrary constant λ [24–26]:

$$(D_x^2 - \lambda) f \cdot f' = 0 \qquad (D_t + 3\lambda D_x + D_x^3) f \cdot f' = 0. \tag{B7}$$

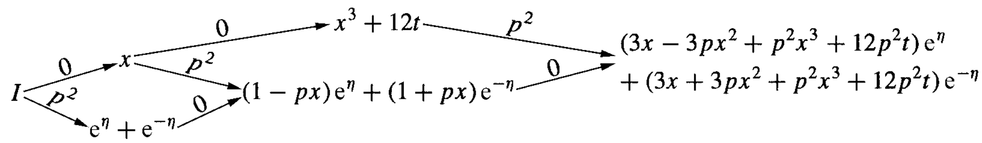
Concerning BT (B7), we have the following [24–26].

Proposition B.3. Let f_0 be a solution of (B2). Suppose that f_1 and f_2 are two solutions of (B2) such that $f_0 \xrightarrow{\lambda_i} f_i$ $(i = 1, 2)$ and $f_j \neq 0$ $(j = 0, 1, 2)$. Then f_{12} defined by

$$f_0 f_{12} = c D_x f_1 \cdot f_2 \qquad (\text{where } c \text{ is a non-zero constant})$$

is a new solution of (B2) which is related to f_1 and f_2 under the BT (B7) with parameters λ_2 and λ_1 respectively.

Now using proposition B.3 and the results above, we have, as an illustrative example,



Therefore $(1-px)\exp(\eta)+(1+px)\exp(-\eta)$ and $(3x-3px^2+p^2x^3+12p^2t)\exp(\eta)+(3x+3px^2+p^2x^3+12p^2t)\exp(-\eta)$ are solutions of the KdV equation (B2), where $\eta = px-4p^3t+\eta^0$; p and η^0 are constants. It is apparent that more solutions of (B2) can be found following this line.

We now turn to the Boussinesq equation

$$3u_{tt}+3(u^2)_{xx}+u_{xxxx}=0. \tag{B8}$$

By the dependent variable transform $u = 2(\ln f)_{xx}$, equation (B8) is written in bilinear form as

$$(D_x^4+3D_t^2)f\cdot f=0. \tag{B9}$$

A BT for (B9) is [24]

$$(D_t+D_x^2)f\cdot f'=0 \qquad (D_x^3-3D_xD_t)f\cdot f'=0. \tag{B10}$$

We represent (B10) symbolically by $f \to f'$. Concerning (B9), we have the following result.

Proposition B.4. Suppose f_0, f_1 and f_{12} are three solutions of (B9); $f_0 \to f_1 \to f_{12} \to f_0$ and $f_0, f_1 \neq 0$. Then there exists an f_2 determined by

$$D_xf_1\cdot f_2=cf_0f_{12} \qquad \text{(where } c \text{ is a non-zero constant)} \tag{B11}$$

such that f_2 is a new solution of (B9) and

$$f_0 \to f_2 \to f_{12}.$$

Proof. First we choose a particular solution $\tilde{f}_2$ from (B11). Thus we have

$$D_xf_1\cdot\tilde{f}_2=cf_0f_{12}.$$

In this case, we have

$$D_x[D_tf_1\cdot\tilde{f}_2+cD_xf_0\cdot f_{12}]\cdot f_1^2=0$$

which implies that

$$D_tf_1\cdot\tilde{f}_2+cD_xf_0\cdot f_{12}=k(t)f_1^2 \tag{B12}$$

where $k(t)$ is a suitable function of t. Now we choose

$$f_2=\tilde{f}_2+f_1\int^t k(t')\,\mathrm{d}t'.$$

It is easily verified that f_2 satisfies (B11) and

$$D_tf_1\cdot f_2+cD_xf_0\cdot f_{12}=0. \tag{B13}$$

Furthermore, a detailed calculation shows that

$$(D_t+D_x^2)f_0\cdot f_2=0 \qquad (D_t+D_x^2)f_2\cdot f_{12}=0$$
$$(D_x^3-3D_xD_t)f_0\cdot f_2=0 \qquad (D_x^3-3D_xD_t)f_2\cdot f_{12}=0.$$

Thus we have completed the proof of proposition B.4. □

As an application of proposition B.4, we can obtain the homogeneous degree polynomial solutions of (B9). To this end, we define the degree of x and t as

$$\deg(x) = 1 \qquad \deg(t) = 2.$$

We seek polynomial solutions of (B9) via the following steps. First choose polynomial seed solutions f_0, f_1 and f_{12} of (B9) such that

$$f_0 \to f_1 \to f_{12} \to f_0$$

and $\deg(f_0) = m$, $\deg(f_1) = n$, $\deg(f_{12}) = l$. Secondly, we find a particular polynomial $\tilde{f}_2$ of degree $m+l-n+1$ such that (B11) holds. From (B12), we know when $m+l$ is not divided by 2, $k(t)$ is a monomial of t of degree $(m+l-n+1-2-n)/2 = (m+l-2n-1)/2$, i.e. $k(t) = kt^{(m+l-2n-1)/2}$ (k is a constant). When $m+l \mid 2$, we can easily deduce that $k(t) = 0$. Now we set

$$f_2 = \tilde{f}_2 \qquad m+l \text{ is not divided by 2}$$
$$f_2 = \tilde{f}_2 + \frac{2k}{m+l-2n+1} t^{(m+l-2n+1)/2} \qquad m+l \text{ is not divided by 2.}$$

Then f_2 is a polynomial solution of (B9) and $f_{12} \to f_0 \to f_2 \to f_{12}$, and we can choose f_{12}, f_0, f_2 as new seed solutions of the next step. In this way, we can obtain a series of polynomial solutions of (B9) step by step. For example, choose $P_0 = 1$, $P_1 = x$, $P_2 = t + \frac{1}{2}x^2$, and we have $P_1 \to P_0 \to P_2 \to P_1$. It is easily verified that $\tilde{P}_4 = x^4 + 4tx^2$ satisfies $D_x P_0 \cdot \tilde{P}_4 = -8P_1P_2$, and $D_t P_0 \cdot \tilde{P}_4 - 8D_x P_1 \cdot P_2 = -8tP_0^2$. Therefore $P_4 = \tilde{P}_4 - 4t^2 P_0 = x^4 + 4tx^2 - 4t^2$ is a solution of (B9) and $P_2 \to P_1 \to P_4 \to P_2$. Furthermore, it can be verified that $P_6 = x^6 + 10x^4 t + 20x^2t^2 + 40t^3$ satisfies $D_x P_1 \cdot P_6 = -10P_2P_4$ and $P_4 \to P_2 \to P_6 \to P_4$. In general, along this line, we can deduce recursively a series of polynomial solutions of (B9). In the following we begin with another recursion process. We have $P_0 \to P_1 \to P_2 \to P_0$. It is easily verified that $Q_2 = t - \frac{1}{2}x^2$ satisfies that $D_x P_1 \cdot Q_2 = P_0 P_2$ and $D_t P_1 \cdot Q_2 - D_x P_0 \cdot P_2 = 0$. Therefore Q_2 is a solution of (B9) and $P_2 \to P_0 \to Q_2 \to P_2$. Furthermore, it can be verified that $Q_5 = x^5 - 20t^2 x$ satisfies $D_x P_0 \cdot Q_5 = 20P_2Q_2$, and $D_t P_0 \cdot Q_5 - 20D_x P_2 \cdot Q_2 = 0$. Therefore Q_5 is also a solution of (B9) and $Q_2 \to P_2 \to Q_5 \to Q_2$. In general, along this line, we can deduce recursively another series of polynomial solutions of (B9). Besides, it is noted that if $f(x,t)$ is a solution of (B9), then $f(-x,t)$, $f(x,-t)$ are also solutions of (B9). Thus more polynomial solutions of (B9) can be found.

Appendix C. Proof of proposition 2.2

First we choose a particular solution F from (2.4). Thus we have

$$D_x f_1 \cdot F = c f_0 f_{12}.$$

In this case, we have, similar to (B12),

$$D_y f_1 \cdot F + \frac{c}{a} D_x f_0 \cdot f_{12} = k_1(t,y) f_1^2 \tag{C1}$$

where $k_1(t,y)$ is a suitable function of t and y. Furthermore, we have

$$\begin{aligned}
0 &= [(D_t + D_x^3 - 3aD_xD_y) f_0 \cdot f_1] f_1 f_{12} - f_0 f_1 (D_t + D_x^3 - 3aD_xD_y) f_1 \cdot f_{12} \\
&= D_t f_0 f_{12} \cdot f_1^2 + \frac{1}{4} D_x^3 f_0 f_{12} \cdot f_1^2 + \frac{3}{4} D_x[(D_x^2 f_0 \cdot f_{12}) \cdot f_1^2 \\
&\quad + f_0 f_{12} \cdot (D_x^2 f_1 \cdot f_1)] - \frac{3a}{2} D_x (D_y f_0 \cdot f_{12}) \cdot f_1^2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3a}{2}D_y(D_x f_0\cdot f_{12})\cdot f_1^2\\
&= D_x\left[\frac{1}{c}D_t f_1\cdot F+\frac{3}{4}D_x^2 f_0\cdot f_{12}-\frac{3a}{2}D_y f_0\cdot f_{12}\right]\cdot f_1^2\\
&\quad+\frac{1}{4c}D_x^3(D_x f_1\cdot F)\cdot f_1^2+\frac{3}{4c}D_x(D_x f_1\cdot F)\cdot(D_x^2 f_1\cdot f_1)\\
&\quad+\frac{3}{2}D_x[(D_x^2 f_0\cdot f_1)\cdot f_1 f_{12}-f_0 f_1\cdot(D_x^2 f_1\cdot f_{12})]\\
&= D_x\left[\frac{1}{c}D_t f_1\cdot F+\frac{3}{4}D_x^2 f_0\cdot f_{12}-\frac{3a}{2}D_y f_0\cdot f_{12}\right]\cdot f_1^2\\
&\quad+\frac{1}{4c}D_x^3(D_x f_1\cdot F)\cdot f_1^2+\frac{3}{4c}D_x(D_x f_1\cdot F)\cdot(D_x^2 f_1\cdot f_1)\\
&\quad+\frac{3}{2}D_x[(D_x^2 f_0\cdot f_{12})\cdot f_1^2-f_0 f_{12}\cdot(D_x^2 f_1\cdot f_1)]\\
&= D_x\left[\frac{1}{c}D_t f_1\cdot F+\frac{9}{4}D_x^2 f_0\cdot f_{12}-\frac{3a}{2}D_y f_0\cdot f_{12}\right]\cdot f_1^2\\
&\quad+\frac{1}{4c}D_x^3(D_x f_1\cdot F)\cdot f_1^2-\frac{3}{4c}D_x(D_x f_1\cdot F)\cdot(D_x^2 f_1\cdot f_1)\\
&= D_x\left[\frac{1}{c}D_t f_1\cdot F+\frac{9}{4}D_x^2 f_0\cdot f_{12}-\frac{3a}{2}D_y f_0\cdot f_{12}+\frac{1}{4c}D_x^3 f_1\cdot F\right]\cdot f_1^2
\end{aligned}
$$

which implies that

$$
D_t f_1\cdot F+\frac{9c}{4}D_x^2 f_0\cdot f_{12}-\frac{3ac}{2}D_y f_0\cdot f_{12}+\frac{1}{4}D_x^3 f_1\cdot F=k_2(t,y)f_1^2 \tag{C2}
$$

where $k_2(t, y)$ is a suitable function of t, y. We can prove that

$$
\begin{aligned}
&D_y\left[\frac{1}{c}D_t f_1\cdot F+\frac{9}{4}D_x^2 f_0\cdot f_{12}-\frac{3a}{2}D_y f_0\cdot f_{12}+\frac{1}{4c}D_x^3 f_1\cdot F\right]\cdot f_1^2\\
&\qquad= D_t\left[\frac{1}{c}D_y f_1\cdot F+\frac{1}{a}D_x f_0\cdot f_{12}\right]\cdot f_1^2
\end{aligned}
$$

which implies $k_{2y} = k_{1t}$. We choose $k(t, y)$ such that $k_t = k_2, k_y = k_1$ and set $f_2 = F + k(t, y)f_1$. It is easily verified that f_2 satisfies (2.4) and

$$
D_y f_1\cdot f_2+\frac{c}{a}D_x f_0\cdot f_{12}=0 \tag{C3}
$$

$$
D_t f_1\cdot f_2+\frac{9c}{4}D_x^2 f_0\cdot f_{12}-\frac{3ac}{2}D_y f_0\cdot f_{12}+\frac{1}{4}D_x^3 f_1\cdot f_2=0. \tag{C4}
$$

Using (C3) and (C4), we can prove

$$
(aD_y+D_x^2)f_0\cdot f_2=0 \qquad (D_t+D_x^3-3aD_xD_y)f_0\cdot f_2=0
$$

and

$$
(aD_y+D_x^2)f_2\cdot f_{12}=0 \qquad (D_t+D_x^3-3aD_xD_y)f_2\cdot f_{12}=0.
$$

Thus we have completed the proof of proposition 2.2.

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